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# Progress in a Trajectory Interpretation of the Klein–Gordon Equation

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A trajectory interpretation is developed for the Klein-Gordon equation in one dimension. The development is couched in a Hamilton-Jacobi representation. Equations of motion are developed. Different trajectories for a given eigenvalue energy are shown to manifest different microstates of the eigenfunction of that particular energy.

## 1. INTRODUCTION

Geometric ray theory is an asymptotic method for describing wave propagation in the short-wavelength limit. As such, it is incomplete for describing wave mechanics in general. Recently, a rigorous ray theory that accounts for finite wavelength has been developed and applied to underwater acoustics, where the index of refraction is to first approximation horizontally stratified (Floyd, 1976, 1984a,b). Rigorous ray theory has been developed in a generalized Hamilton-Jacobi representation. The rays of rigorous ray tracing obey equations of motion different than Snell's law for geometric ray theory (Floyd, 1984a), penetrate into the classically forbidden region beyond the WKB turning (internal refraction) points (Floyd, 1984a), and are not in general normal to the surfaces of constant Hamilton's characteristic function, albeit these surfaces remain transversals of the ray (Floyd, 1986a). As the horizontal ducts of underwater acoustics correspond to one-dimensional potential wells in quantum mechanics, a trajectory interpretation has been concurrently developed for one-dimensional time-independent nonrelativistic quantum mechanics (Floyd, 1982a,b, 1984b, 1986c). While the trajectories of this interpretation have shown the attributes of rays of rigorous ray theory, this trajectory interpretation has also shown that the Schrödinger wave function is not an exhaustive description of

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nature, because different trajectories in phase space have the same action quantization and describe microstates for the wave function of the corresponding energy eigenvalue (Floyd, 1982b, 1986c). Furthermore, these trajectories obey equations of motion different than those of Bohm (Bohm, 1952; Floyd, 1982b).

Rigorous ray theory has been developed so far to tackle one-dimensional problems of horizontally stratified ducts of underwater acoustics. While the application to problems where separation of variables is straightforward, the application to higher order dimensions is regrettably nontrivial in general. A general mathematical treatment for higher order dimensions has yet to be developed. Nevertheless, a trajectory interpretation, even in only one dimension in configuration space (i.e., two dimensions in phase space), has revealed and resolved many interesting issues in the foundations of nonrelativistic quantum theory.

The objective herein is to extend the trajectory interpretation of nonrelativistic quantum mechanics, albeit limited to one dimension in configuration space, to spinless bosons described by the Klein-Gordon equation for the time-independent case. The equations of motion for trajectories in phase space are developed from a phenomenological Hamilton-Jacobi equation for relativistic continuous quantum motion for spinless bosons. Ramifications are discussed, and the confinement paradox (Bjorken and Drell, 1964; Klein, 1929) of the Klein-Gordon equation is shown to manifest itself in the trajectory interpretation. One dimension is sufficient for developing most issues herein, so four-vector notation is not used.

The rest of this exposition is organized with Sections 2-4 committed to mainly a mathematical development and Sections 5-7 committed to mainly an interpretational development. In Section 2, we present a Hamilton-Jacobi representation for relativistic quantum continuous motion. In Section 3, the various trajectories for a specified energy eigenvalue are shown to specify microstates that have the same quantization for the action variable. In Section 4, we develop the equations of motion for relativistic continuous quantum motion. In Section 5, the confinement paradox is discussed. In Section 6, an interpretation for the superluminal character of the trajectory is presented. In Section 7, quantum measurements with respect to a trajectory representation are discussed.

## 2. THE HAMILTON-JACOBI EQUATION

We assume a time independence. Let us consider the one-dimensional, $x$, phenomenological Hamilton-Jacobi equation for relativistic continuous quantum motion for spinless particles given by

$$(\partial W/\partial x)^2-(1/c^2)(E^2-m^2c^4-2EV+V^2)=-\tfrac{1}{2}\hbar^2\langle W;x\rangle \qquad (1)$$

where $W$ is Hamilton's characteristic function for relativistic continuous quantum motion, $c$ is the speed of light, $E$ is energy, $m$ is the rest mass, $V$ is the potential and a function of $x$, and $\hbar$ is Planck's constant divided by $2\pi$. The Schwarzian derivative is defined by

$$\langle W; x\rangle \equiv \frac{\partial^3 W/\partial x^3}{\partial W/\partial x} - \frac{3}{2}\left[\frac{\partial^2 W/\partial x^2}{\partial W/\partial x}\right]^2$$

The solution to equation (1) is given by

$$\partial W/\partial x = (2m)^{1/2}/(a\phi^2 + b\theta^2 + d\phi\theta) \tag{2}$$

where $a$ and $b$ are positive-definite constants, $d$ is a constant such that $d^2 < 4ab$, and $\phi$ and $\theta$ form a set of independent solutions to the Klein-Gordon equation for $\psi$

$$-c^2\hbar^2\,\partial^2\psi/\partial x^2 + (m^2c^4 - E^2 + 2EV - V^2)\psi = 0 \tag{3}$$

The solutions $\phi$ and $\theta$ are scaled such that the Wronskian $\mathscr{W}$ for equation (3) is given by

$$\mathscr{W}(\phi, \theta) = (1/\hbar)[2m/(ab - d^2/4)]^{1/2}$$

The substitution of the equation (2) as the solution into equation (1), the Hamilton-Jacobi equation, leads to equation (3), the Klein-Gordon equation, which itself is also a phenomenological equation.

We may choose a set of independent solutions such that $d = 0$ (Floyd, 1986b). In such case, the solution for the Hamilton-Jacobi equation may be represented by

$$\partial W/\partial x = (2m)^{1/2}/(a\phi^2 + b\theta^2) \tag{2}$$

In one dimension, the solutions $\phi$ and $\theta$ may always be real (Landau and Lifshitz, 1958). Thus, the conjugate momentum $\partial W/\partial x$ is always real, even in the classically forbidden zone.

In the nonrelativistic limit, we let $E = E' + mc^2$, where $mc^2 \gg E'$, and equation (1) may be represented by

$$\tfrac{1}{2}\hbar^2\langle W; x\rangle + (\partial W/\partial x)^2 = 2m(E' - V) + O(c^{-2}) \tag{3}$$

which is consistent to order $c^{-1}$ with the nonrelativistic Hamilton-Jacobi equation for quantum continuous motion (Floyd, 1984b, 1986b).

## 3. QUANTIZATION AND MICROSTATES

Let us examine the action variable $J$ for a bound state as given by

$$J = \oint \frac{\partial W}{\partial x}\,dx = \hbar \oint \frac{(ab - d^2/4)^{1/2}\mathscr{W}(\phi, \theta)}{a\phi^2 + b\theta^2 + d\phi\theta}\,dx \tag{4}$$

Equation (4) is now in the same form as the equation for the action variable for nonrelativistic continuous quantum motion (Floyd, 1986c), where it was shown that $J=2Nh$, where $N$ is the order of the eigenfunction and $h$ is Planck's constant. Hence $J$ is quantized in accordance with the order of the eigenfunction as specified by $E$. The quantization of $J$ is independent of the coefficients $a$, $b$, and $d$. But these coefficients specify different conjugate momenta as a function of $x$ and, so, different trajectories. Thus, each trajectory represents a different microstate for this action variable quantization.

Let us consider a new independent set $(\zeta, \xi)$ of solutions for the Klein-Gordon equation in one dimension, which reduces to a Helmholtz equation, represented by (Floyd, 1986b, 1986c)

$$\begin{aligned}\zeta &= (a\phi^2+b\theta^2+d\phi\theta)^{1/2} \\ &\quad\times\cos\left[\int^x (ab-d^2/4)^{1/2}\mathcal{W}(\phi,\theta)(a\phi^2+b\theta^2+d\phi\theta)^{-1}\,dx\right] \\ &= [a-d^2/(4b)]^{1/2}\phi \end{aligned} \tag{5}$$

and

$$\begin{aligned}\xi &= (a\phi^2+b\theta^2+d\phi\theta)^{1/2} \\ &\quad\times\sin\left[\int^x (ab-d^2/4)^{1/2}\mathcal{W}(\phi,\theta)(a\phi^2+b\theta^2+d\phi\theta)^{-1}\,dx\right] \\ &= b^{1/2}\theta+d\phi/(2b^{1/2})\end{aligned}$$

For this new set, the conjugate momentum $p$ for relativistic continuous quantum motion and the Wronskian $\mathcal{W}$ may be given as

$$p=(2m)^{1/2}/(a\phi^2+b\theta^2+d\phi\theta)=(2m)^{1/2}/(\zeta^2+\xi^2) \tag{6}$$

and

$$\mathcal{W}(\zeta,\xi)=(ab-d^2/4)^{1/2}\mathcal{W}(\phi,\theta)$$

Equation (6) confirms that we may always choose a set of solutions such that $d=0$.

Let us set $\zeta=\alpha\phi$ in equation (5), where $\alpha$ is a coefficient. Then equation (5) becomes an identity with $\alpha=[a-d^2/(4b)]^{1/2}$. Hence, the coefficients $a$, $b$, and $d$ only effect the normalization of the wave function and do not change the predictions of quantum measurement of an observable (i.e., $\int\psi^\dagger\Lambda\psi/\int\psi^\dagger\psi$, where $\Lambda$ is the operator for the observable). Nevertheless, the coefficients determine different trajectories in phase space. The different trajectories represent different microstates of the same wave function. For a given energy, the different trajectories may be specified by the coefficients

$a$ and $b$, since a set of solutions may be chosen for which $d = 0$. Examples of different nonrelativistic orbital trajectories in phase space for the same eigenvalue $E$ have been presented by Floyd (1982b). The coefficients $a$ and $b$ are hidden variables, along with the constant of the motion $E$, that specify the various microstates, which manifest different trajectories, of a particular wave function.

## 4. EQUATIONS OF MOTION

While the equations of motion may be developed in a representation described by the set of independent solutions $(\phi, \theta)$, these solutions are not always known. For convenience, let us introduce a modified potential $U$ specified by

$$U = \frac{2[V - V^2/E + \frac{1}{2}\hbar^2\langle W; x\rangle]}{1 \pm [1 - 2V/E + V^2/E - \frac{1}{2}\hbar^2\langle W; x\rangle]^{1/2}}$$

We choose the plus sign in the above equation so that $U \to V$ as $E \to \infty$. An alternate specification for $U$ is given by

$$\frac{1}{2}\frac{(E-U)\,\partial^2 U/\partial x^2 - (\partial U/\partial x)^2}{E^2 - m^2c^4 - 2EU + U^2} + \frac{5}{4}\left[\frac{(E-U)\,\partial U/\partial x}{E^2 - m^2c^4 - 2EU + U^2}\right]^2 + \frac{1}{\hbar^2 c^2}[2E(U-V) - U^2 + V^2] = 0 \tag{7}$$

From equation (7), $U$ is dependent on $E$. Since equation (7) is nonlinear, $U$ has nonlinear critical points, which for this case are classified as nodal points, since

$$U \to E \pm mc^2 \qquad \text{as} \qquad x \to \infty$$

for bound state energies. The solution for $U$ may be given in closed form, where known, by

$$U = E \pm \left[m^2c^4 + \frac{2mc^2}{(a\phi^2 + b\theta^2 + d\phi\theta)^2}\right]^{1/2} \tag{8}$$

and therefore $U$ is dependent upon the particular trajectory [i.e., $U = U(x, E, a, b)$ as a solution set $(\phi, \theta)$ can always be chosen such that $d = 0$].

The Hamilton–Jacobi equation may now be expressed in terms of $U$ as

$$(\partial W/\partial x)^2 - (1/c^2)(E^2 - m^2c^4 - 2EU + U^2) = 0 \tag{9}$$

In this representation, the Hamilton–Jacobi equation appears as a first-order nonlinear differential equation, which may be directly integrated, albeit $U$ is dependent upon the trajectory. As $E$ appears intrinsically in $U$, $E$ is no longer a first integral of the motion, although it remains a constant of the motion. From the integral form of the Hamilton–Jacobi equation,

$$S(x, t, E, a, b) = W(x, E, a, b) - Et$$

where $S$ is Hamilton's principal function and $t$ is time and where the trajectory dependence upon the hidden variables $a$ and $b$ has been made explicit, the equation of motion for the trajectory may be expressed by

$$\dot{x} = \frac{c(E^2 - m^2c^4 - 2EU + U^2)^{1/2}}{(E - U)(1 - \partial U/\partial E)} \tag{10}$$

From equations (7) and (10), we may numerically integrate concurrently the trajectory and $U$ for those cases for which we do not know the closed form of the solution set $(\phi, \theta)$.

We note that

$$\dot{x} \neq \frac{\partial W/\partial x}{\{m^2 + [(\partial W/\partial x)/c]^2\}^{1/2}}$$

so that, in general, the conjugate momentum is not the relativistic mechanical momentum, i.e., $\partial W/\partial x \neq m\dot{x}[1 - (\dot{x}/c)^2]^{1/2}$. If $U$ should be independent of $E$, then the conjugate and mechanical momenta are the same. This implies that in more than one dimension the trajectory is not necessarily normal to the transversal planes of constant $W$ (Floyd, 1982b, 1984a).

Since equation (7) is nonautonomous, $U$ is a nonlocal potential (physically, it is trajectory-dependent), and the hidden variables $a$ and $b$ that specify the trajectories (microstates) are classified as nonlocal hidden variables.

## 5. THE CONFINEMENT PARADOX

In this section, let us consider a potential well where $V(x)$ increases without limit as $|x| \to \infty$. This situation is the well-known confinement paradox (Klein, 1929; Bjorken and Drell, 1964). For such potentials, the solution for the Klein–Gordon equation becomes oscillatory again far outside the well. This effect is exacerbated by making the potential more tightly binding. The forbidden region is separated from the oscillatory (allowed) region in configuration space by WKB turning points $x_t$, where $V(x_t) = E \pm mc^2$. This paradox arises as a result of incorporating solutions of negative energy and negative rest mass into the Klein–Gordon equation. The well-known resolution to the paradox is that the negative-energy solutions represent antiparticles moving backward in time.

In a trajectory interpretation, the trajectories penetrate the forbidden region consistent with equation (2′). In the less-confined potential well case where the wave function does not become oscillatory again, the turning points recede to infinity, where $\partial W/\partial x$ becomes zero asymptotically (Floyd, 1982b).

But for the confinement problem, $\partial W/\partial x$ is never zero, and there is no turning point. The trajectories pass right through the forbidden region connecting the "particle" oscillatory region with the "antiparticle" region. Here we have a metastable condition represented on a time-independent trajectory where particles and their antiparticles are represented on the same trajectory without any apparent annihilation and creation points. The Klein–Gordon equation describes bosons, and all known spinless bosons are unstable. Hence, the trajectory through the forbidden region between the oscillatory regions represents a tunneling by a metastable state.

Antiparticles call for the use of the other square root of equation (8). As $\phi$ and $\theta$ are real, the nodal form of the critical point behavior of $U$ effectively confines $U$ to two disjointed domains in energy, $U \leq E - mc^2$ for particles and $U \geq E + mc^2$ for antiparticles. A nodal point in equation (7), where $U = E \pm mc^2$, renders an opportunity to shift roots between particle and antiparticle solutions in equation (8) while maintaining a continuous $\partial U/\partial x$. But a nodal point in equation (7) implies a nodal point in equation (9), where $\partial W/\partial x = 0$, which concurrently describes a turning point.

There exists a mechanism for producing on the trajectory a point that demarcates the particle side from the antiparticle side. Let us endeavor to induce a nodal singularity in $U$ at some finite $x_0$. Such a node would function as a branch point singularity for shifting roots in equation (9) between the particle and antiparticle roots while maintaining a smooth trajectory. Let $U$ have a local maximum at $x_0$ such that $U$ may be approximated in a neighborhood sufficiently close to $x_0$ by

$$U = E - mc^2 - \varepsilon - A(x - x_0)^2, \qquad 0 < \varepsilon \ll 1 \tag{11}$$

where $A$ is a real, nonnegative coefficient for the quadratic term and $\varepsilon$ is positive and may be made infinitesimally small. If equation (11) is substituted into equation (7), a quartic equation in $A$ is generated, for which one solution has the form

$$A = \frac{2\varepsilon}{\hbar^2 c^2}[E - mc^2 - V(x_0)][E + mc^2 - V(x_0)] \tag{12}$$

while the other three solutions for $A$ contain the factor $(x - x_0)$ to some order and therefore are invalid solutions for the coefficient of $(x - x_0)^2$. As $A$ contains the factor $\varepsilon$, $A$ may be made infinitesimal and we may get arbitrarily close to inducing a branch point singularity. From equation (12), the coefficient $A$ can only be positive for $x_0$ in one of the oscillatory regions.

So we cannot induce a branch point singularity in the forbidden region where the trajectory is tunneling between the particle and antiparticle regions, and any smooth transition must occur in one of the oscillatory regions. Of course, we could allow a shift between particle and antiparticle solutions to equation (10) in the forbidden region if we accept a discontinuity in momentum of the trajectory in phase space.

## 6. TRAJECTORY VELOCITY

Here, we consider a loosely bound state such that $V(x) < E + mc^2$. Then the only oscillatory region for the wave function is located in the potential well. The turning points for the trajectory recedes to infinity (i.e., $x_t \to \pm\infty$). At the turning point, $\partial W/\partial x \to 0$ as $x \to \pm\infty$. But concurrently, at the turning point, equation (10) must be evaluated by l'Hôpital's limiting process to determine that $\dot{x} \to \pm\infty$ as $x \to \infty$ where $\dot{x}$ becomes superluminal. Therefore, the trajectory cannot be interpreted to imply the transfer of material, either mass or energy [the conjugate momentum $\partial W/\partial x$ and its Schwarzian derivative are inputs to energy in accordance with equation (1)]. This phenomenon is best exhibited in higher dimensions. As noted in Section 3, the trajectory is generally not normal to the surfaces of constant $W$, albeit these surfaces are still transversals of the trajectory. At the infinite turning point, the trajectory is embedded in the plane of its transversal and its velocity becomes infinite while the conjugate momentum goes to zero (Floyd, 1984a). The trajectory velocity is then a phenomenological ray velocity.

## 7. QUANTUM MEASUREMENTS

In one dimension and for time independence, both the Hamilton–Jacobi equation for continuous relativistic quantum motion, equation (1), and the Klein–Gordon equation are phenomenological descriptions. Is one more fundamental than the other? The answer depends upon one's interpretation of quantum measurement.

If we could somehow measure, determine, or prescribe the set of hidden variables $a$ and $b$ that determine the particular trajectory or microstate of a wave function with eigenvalue energy $E$, then we would know the relativistic continuous quantum motion for a particle, including excursions into the classically forbidden regions. While a unique solution (trajectory) for equation (1) is determined by the initial conditions of $\partial W/\partial x$ and $\partial^2 W/\partial x^2$ at $x_0$, we may also determine the trajectory within limitations of the multiple trajectory problem from boundary conditions from knowledge of the times at which the trajectory passes through three or more spatial locations. The hidden variables are sufficient to determine a generator of the motion, Hamilton's characteristic function, for a particular microstate. In this interpretation, a Hamilton–Jacobi representation is more fundamental than the Klein–Gordon representation.

If one asserts a positivistic interpretation of quantum mechanics and alleges that the individual trajectories are only phenomenological and devoid of any physical meaning, then the Hamilton–Jacobi representation is still equivalent to the Klein–Gordon representation, as the trajectories (microstates) describe the same wave function (Floyd, 1986b) and the quantum measurement for an observable $\lambda$ with operator $\Lambda$ is still given by

$$\lambda = \int \psi^\dagger \Lambda \psi \Big/ \int \psi^\dagger \psi$$

Of course, in this positivistic interpretation, we have discarded the information rendered by the hidden variables $a$ and $b$, which specify the particular trajectory. And if observations are sufficient to determine a specific trajectory, either by boundary values or initial conditions, in a nonclassical limit, then a positivistic interpretation would not attempt to specify the trajectory for continuous quantum motion, but rather would attribute the observations to an observed sample of the ensemble of quantum probabilities of a system in the nonclassical limit.

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